

the complementary relations of the relative entropy of coherence in quantum information processing

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ABSTRACT

We relate relative entropy of coherence to quantum dense coding, teleportation and quantum phase transition. We first give an uncertainty-like expression relating local quantum coherence to the capacity of optimal dense coding for bipartite system. Further, the relation between local quantum coherence and teleportation fidelity for two-qubit system is given.

Introduction

Quantum coherence, which arises from quantum superposition, is a fundamental feature of quantum mechanics, and it is also an essential ingredient in quantum information and computation¹. Furthermore, in some emergent fields, such as quantum metrology^{2,3}, nanoscale thermodynamics⁴⁻⁸ and quantum biology⁹⁻¹², quantum coherence plays a central role.

Recently, the information-theoretic quantification of quantum coherence is a successful application of quantum resource theory¹³. Baumgratz et al. proposed the basic notions of incoherent states, incoherent operations and a series of necessary conditions any measures of coherence should satisfy. In this sense, coherence is defined as the resource relative to the set of incoherent operations. According to the postulates in the framework, relative entropy of coherence¹³, l_1 -norm of coherence¹³ and other coherence measures¹⁴⁻¹⁸ have been put forward. Based on coherence measures, the relations between quantum coherence and other resources^{14,19,20}, the complementarity relations of quantum coherence²¹ and other properties of quantum coherence^{22,23} have been investigated. Mainly due to the interest aroused by the resource theory of quantum coherence, there have been several attempts at understanding the role of coherence as a resource for quantum protocols. For example, in the incoherent quantum state merging, which is the same as standard quantum state merging up to the fact that one of the parties has free access to local incoherent operations only and has to consume a coherent resource for more general operations, the entanglement-coherence sum is non-negative, and no merging procedure can gain entanglement and coherence at the same time²⁴. Perfect incoherent teleportation of an unknown state of one qubit is possible with one singlet and two bits of classical communications²⁵. Here, the incoherent teleportation is the same as standard teleportation up to the fact that local operations and classical communications are replaced by local incoherent operations and classical communications. Furthermore, the notion of coherence as a symmetry relative to a group of translations naturally shows up in the context of quantum speed limits because the speed of evolution is itself a measure of asymmetry relative to time translations²⁶.

As we know, both quantum coherence and entanglement relate to quantum superposition. Moreover, many quantum information protocols, such as dense coding and teleportation, would be impossible without the assistance of entanglement. Therefore, bridged by entanglement, we want to directly relate quantum coherence with the protocols of quantum information. Specifically, we want to give the quantitative relation between quantum coherence and the dense coding capacity or teleportation fidelity.

In the present work, we will establish the relation between quantum coherence and the optimal dense coding capacity, and also relate quantum coherence to teleportation fidelity. Here, quantum coherence is measured by relative entropy of coherence.

Results

Relating quantum coherence to optimal dense coding and teleportation

In this section, we investigate the relation between quantum coherence and optimal dense coding, and that between quantum coherence and teleportation.

The definition of relative entropy of coherence C_{re} ¹³ is

$$C_{\text{re}}(\rho) = \min_{\delta \in \mathcal{I}} S(\rho \| \delta), \quad (1)$$

where $S(\rho\|\delta) = \text{tr}(\log_2 \rho - \log_2 \delta)$ is the relative entropy. \mathcal{I} is the set of all incoherent states and all density operators $\delta \in \mathcal{I}$ are of the form¹³

$$\delta = \sum_{i=1}^d \delta_i |i\rangle\langle i|, \quad (2)$$

$\{|i\rangle\}_{i=1,\dots,d}$ is a particular basis of the d -dimensional Hilbert space \mathcal{H} . In the definition of relative entropy of coherence, the minimum is attained if and only if $\delta = \rho^{\text{diag}}$ with ρ^{diag} being the diagonal part of ρ . C_{re} satisfies the four postulates given in Ref. 13 which are the conditions that a measure of quantum coherence should suffice. Based on the definition, we can establish the relation between quantum coherence and optimal dense coding.

Relating quantum coherence to optimal dense coding

For a bipartite quantum state ρ_{AB} on two d -dimensional Hilbert spaces $\mathcal{H}_A^d \otimes \mathcal{H}_B^d$ and $\rho_B = \text{tr}_A(\rho_{AB})$ being the reduced density matrix of the subsystem B , we have the following theorem.

Theorem 1 The sum of the optimal dense coding capacity of the state ρ_{AB} and quantum coherence of the reduced state ρ_B is always smaller than $2\log_2 d$, i.e.

$$\chi(\rho_{AB}) + C_{\text{re}}(\rho_B) \leq 2\log_2 d, \quad (3)$$

where $\chi(\rho_{AB})$ is the optimal dense coding capacity of the state ρ_{AB} .

Proof. The d^2 signal states generated by mutually orthogonal unitary transformations with equal probabilities will yield the maximal χ ²⁸. The mutual orthogonal unitary transformations are given as

$$U_{m,n}|j\rangle = \exp\left(i\frac{2\pi}{d}mj\right)|j+n(\text{mod } d)\rangle, \quad (4)$$

where integers m and n range from 0 to $d-1$. The ensembles generated by the unitary transformations with equal probabilities $p_{m,n}$ can be denoted as $\mathcal{E}^* = \{(U_{m,n} \otimes I_d)\rho(U_{m,n}^\dagger \otimes I_d); p_{m,n} = 1/d^2\}_{m,n=0}^{d-1}$. The average state of the ensembles is

$$\overline{\rho_{AB}^*} = \frac{1}{d^2} \sum_{m,n} (U_{m,n}^A \otimes I_d^B) \rho (U_{m,n}^{A\dagger} \otimes I_d^B). \quad (5)$$

Here, I_d^B is the d -dimensional identity matrix in the subsystem B . Accordingly, the capacity of the optimal dense coding can be given as²⁸

$$\chi(\rho_{AB}) = S(\overline{\rho_{AB}^*}) - S(\rho_{AB}). \quad (6)$$

Based on the results given in Ref. 28, $\overline{\rho_{AB}^*} = I_d^A \otimes \rho_B/d$ and

$$S(\overline{\rho_{AB}^*}) = -\text{tr}(\overline{\rho_{AB}^*} \log_2 \overline{\rho_{AB}^*}) = -\text{tr}(I_d) \text{tr}\left(\frac{\rho_B}{d} \log_2 \frac{\rho_B}{d}\right) = S(\rho_B) + \log_2 d. \quad (7)$$

For the reduced state ρ_B of the subsystem B , $C_{\text{re}} = S(\rho_B^{\text{diag}}) - S(\rho_B)$, and $S(\rho_B^{\text{diag}}) \leq \log_2 d$. Therefore, $C_{\text{re}}(\rho_B) \leq \log_2 d - S(\rho_B)$, from which we have

$$C_{\text{re}}(\rho_B) + S(\rho_B) \leq \log_2 d. \quad (8)$$

Now, we consider the sum of the optimal dense coding capacity and quantum coherence of the subsystem B

$$\chi(\rho_{AB}) + C_{\text{re}}(\rho_B) = S(\rho_B) + \log_2 d - S(\rho_{AB}) + C_{\text{re}}(\rho_B) \leq \log_2 d + \log_2 d - S(\rho_{AB}) \leq 2\log_2 d, \quad (9)$$

where the first inequality is attained because of the fact given in Eq. (8), and the second inequality is obtained because $S(\rho_{AB}) \geq 0$. This completes the proof.

The inequality given in Eq. (3) indicates that the larger local quantum coherence, the smaller capacity of the optimal dense coding. In other words, if the system AB is used to perform dense coding as much as possible, quantum coherence of the subsystem B would pay for the dense coding capacity of the whole system. The physical reason is that dense coding is based on entanglement, and would be impossible without the assistance of entangled states. The results given in Ref. 20 show that entanglement of the whole system and quantum coherence of a subsystem are complementary to each other - an increase in one leads to a corresponding decrease in the other. For example, for a Bell state, an incoherent state of the subsystem B will

be acquired if qubit A is traced over. On the contrary, creating a superposition on subsystem to have maximum coherence on it will exclude entanglement between subsystems.

For the particular case that the shared entangled state is the Bell state, $\chi(\rho_{AB}) = 2$ and $C_{\text{re}}(\rho_B) = 0$, and the sum of them equals to 2, which just equals to the right hand of Eq. (3).

In Ref. 25, the task of incoherent quantum state merging is introduced and the amount of resources needed for it is quantified by an entanglement-coherence pair. It is found that the entanglement-coherence sum is non-negative, in other words, no merging procedure can gain entanglement and coherence at the same time. From the results given in this paper, it is found the sum of the optimal dense coding capacity and quantum coherence are upper bounded by a definite value. That is, there is a trade-off between the dense coding capacity and quantum coherence. It should be noted that dense coding is based on entanglement, and the former would be impossible when the latter is absent. In this sense, the result given in Eq. (3) is consistent with those presented in Ref. 25.

The relation between quantum coherence and dense coding has been given in Eq. (3), and in the following, we relate quantum coherence to teleportation.

Relating quantum coherence to teleportation

For an arbitrary two-qubit mixed state ρ_{AB} , and $\rho_A = \text{tr}_B(\rho_{AB})$ being the reduced state of the subsystem A , we have the following theorem.

Theorem 2 For any two-qubit state

$$h\left(\frac{1 + \sqrt{1 - [3F(\rho_{AB}) - 2]^2}}{2}\right) + C_{\text{re}}(\rho_A) \leq 1, \quad (10)$$

where $h(x) = -x \log_2 x - (1-x) \log_2 (1-x)$, $F(\rho_{AB})$ being the teleportation fidelity of the state ρ_{AB} and $C_{\text{re}}(\rho_A)$ denoting quantum coherence of the subsystem A . Here, we just consider the case that the state ρ_{AB} is useful for teleportation, which means $F(\rho_{AB}) \geq 2/3$.

Proof. In the proof, the subscripts are omitted in the case that it does not cause confusion. For a two-qubit state, the relation between the teleportation fidelity $F(\rho)$ and negativity $N(\rho)$ is $3F(\rho) - 2 \leq N(\rho)$ ²⁹, while negativity is related to concurrence $C(\rho)$ as $N(\rho) \leq C(\rho)$ ³⁰. Combining two relations, one can obtain $3F(\rho) - 2 \leq N(\rho) \leq C(\rho)$. $F(\rho) \geq 2/3$ leads to all of them being larger than 0, so the square of them also obey the rules, i.e. $[3F(\rho) - 2]^2 \leq N^2(\rho) \leq C^2(\rho)$. Subsequently, the following expression exists

$$\frac{1 + \sqrt{1 - [3F(\rho) - 2]^2}}{2} \geq \frac{1 + \sqrt{1 - N^2(\rho)}}{2} \geq \frac{1 + \sqrt{1 - C^2(\rho)}}{2} \geq \frac{1}{2}. \quad (11)$$

The last inequality can be acquired based on the fact that concurrence $C(\rho)$ for two-qubit state runs from 0 to 1.

As known to all, $h(x)$ is a monotonically decreasing function in the interval $[1/2, 1]$, thus one can obtained

$$h\left(\frac{1 + \sqrt{1 - [3F(\rho) - 2]^2}}{2}\right) \leq h\left(\frac{1 + \sqrt{1 - N^2(\rho)}}{2}\right) \leq h\left(\frac{1 + \sqrt{1 - C^2(\rho)}}{2}\right) = E_F(\rho), \quad (12)$$

where $E_F(\rho)$ is the entanglement of formation of the state ρ_{AB} .

For any bipartite state ρ_{AB} , entanglement of formation and quantum coherence obey the relation²⁰

$$E_F(\rho_{AB}) + C_{\text{re}}(\rho_A) \leq \log_2 d_A. \quad (13)$$

Combining the Eqs. (12) with (13), it is easy to find that, for two-qubit state, we complete the proof.

The inequality given in Eq. (10) indicates that the larger the teleportation fidelity, the less local quantum coherence. That is to say, quantum coherence of the subsystem should pay for teleportation fidelity of the whole system. The reason for this result is that teleportation relies on entanglement. However, quantum coherence of the subsystem and entanglement of the whole system are complementary to each other.

For the particular case that the Bell state is utilized to perform teleportation, $F(\rho_{AB}) = 1$ leads to $h\left(\frac{1 + \sqrt{1 - [3F(\rho_{AB}) - 2]^2}}{2}\right) = 1$ while $C_{\text{re}}(\rho_A) = 0$. Thus, $h\left(\frac{1 + \sqrt{1 - [3F(\rho_{AB}) - 2]^2}}{2}\right) + C_{\text{re}}(\rho_A)$ equals to 1.

As proved in Ref. 20, the relative entropy of coherence is unitary invariant by using the different bases, the results given in Eqs. (3) and (10) hold for all local bases.

From the results given in Eqs. (3) and (10), it is found that there is trade-off between local quantum coherence and the optimal dense coding capacity or the teleportation fidelity. In general, the relation among coherence, discord and entanglement has been given by use of quantum relative entropy, where quantum coherence is found to be a more ubiquitous manifestation quantum correlations¹⁹. Furthermore, for two-qubit states with maximally mixed marginals, the pairwise correlations between local observables are complementary to the coherence of the product bases they define²⁷.

Discussion

In this paper, we relate relative entropy of coherence to quantum dense coding, teleportation and quantum phase transitions. Firstly, we establish an inequality between the optimal dense coding capacity of a bipartite system and local quantum coherence. The inequality indicates that smaller local quantum coherence should be responsible for larger capacity of optimal dense coding. Secondly, an inequality between teleportation fidelity and local quantum coherence for a two-qubit system is given. From the inequality, it is found that the larger the teleportation fidelity, the smaller local quantum coherence. Our results in this paper give a clear quantitative analysis and operational connections between quantum coherence and concepts in quantum information science.

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F.P. and Z.L. initiated the research project and established the main results under the guidance of L.Q. F.P. wrote the manuscript and all authors reviewed the manuscript.

Additional information

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